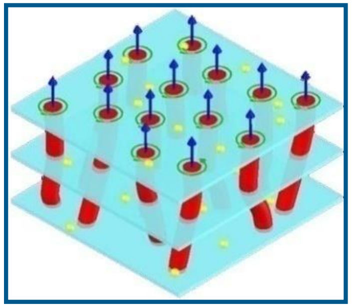
UNIVERSITÉ DE GENÈVE
FACULTÉ DES SCIENCES

Superconductivity and its applications

Lecture 3



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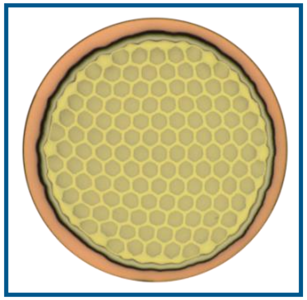


Previously, in lecture 2

Ginzburg-Landau Theory of Superconductivity

A case of order-disorder transition

$$F_s = F_n + \text{Condensation energy} + \text{Kinetic energy} + \text{Field energy}$$

In the G-L theory $n_s(\vec{r}) = |\psi(\vec{r})|^2$ *is the order parameter*

$\psi(\vec{r}) = |\psi(\vec{r})| e^{i\phi}$ *is a complex function*

$$F_S(\vec{r},T) = F_N(\vec{r},T) + \alpha\,|\psi|^2 + \frac{\beta}{2}\,|\psi|^4 + \frac{1}{2m^*}\left|\left(-i\hbar\vec{\nabla} - \frac{e^*}{c}\vec{a}\right)\psi\right|^2 + \frac{h^2}{8\pi}$$

$\alpha = \alpha_1 (T - T_c)$ *and* $\beta = \beta_0$

Previously, in lecture 2

Ginzburg-Landau Theory of Superconductivity

Energy $F_S(T) = \int_V d\vec{r}\; F_S(\vec{r}, T)$

$$F_S(T) = \int_V d\vec{r} \left[F_N(\vec{r}, T) + \alpha\, |\psi|^2 + \frac{\beta}{2}\, |\psi|^4 + \frac{1}{2m^*} \left| \left(-i\hbar\vec{\nabla} - \frac{e^*}{c}\vec{a} \right) \psi \right|^2 + \frac{h^2}{8\pi} \right]$$

Minimization procedure $\Rightarrow$ **variations of ψ, ψ^*, and a**

$$\delta F_S = F_S\left(\psi + \delta\psi,\; \psi^* + \delta\psi^*,\; a + \delta a\right) - F_S\left(\psi, \psi^*, a\right)$$

$$\delta F_S = \int_V d\vec{r} \left[(\qquad)\delta\psi + (\qquad)\delta\psi^* + (\qquad)\delta\vec{a} \right]$$

And set $\frac{\delta F_S}{\delta\psi} = 0 \qquad \frac{\delta F_S}{\delta\psi^*} = 0 \qquad \frac{\delta F_S}{\delta a} = 0$

Previously, in lecture 2

The two Ginzburg-Landau equations

$$\frac{\delta F_S}{\delta \psi} = 0 \qquad \frac{\delta F_S}{\delta \psi^*} = 0 \quad \Rightarrow$$

$$\alpha\ \psi + \beta\ |\psi|^2 \psi + \frac{1}{2m^*}\left(-i\hbar\vec{\nabla} - \frac{e^*}{c}\vec{a}\right)^2 \psi = 0$$

1st G-L equation

$$\frac{\delta F_S}{\delta a} = 0 \quad \Rightarrow$$

$$\vec{j} = \frac{e^*\hbar}{2m^* i}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) - \frac{e^{*2}}{m^* c}|\psi|^2\,\vec{a}$$

2nd G-L equation

Previously, in lecture 2

Resolution of the 1st G-L equation in special cases

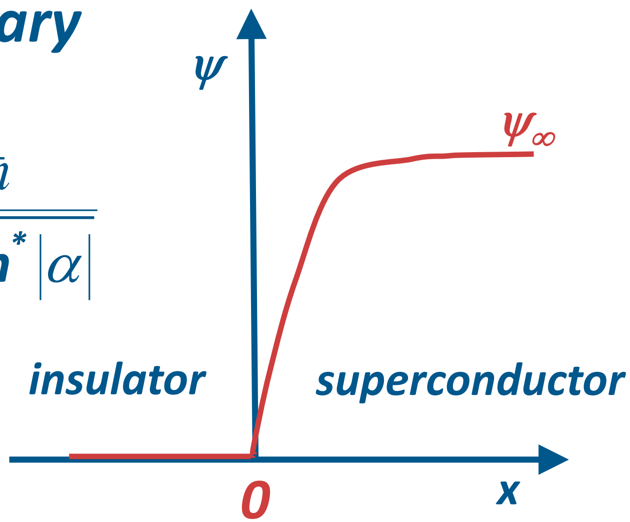
- **Zero-field case deep inside superconductor**

From thermodynamics

$$|\psi| = \left(\frac{\alpha_1}{\beta_0}\right)^{\frac{1}{2}} (T_c - T)^{\frac{1}{2}} \qquad F_S - F_N = -\frac{1}{2}\frac{\alpha^2}{\beta} = -\frac{H_c^2}{8\pi}$$

- **Zero-field case near superconductor boundary**

$$\psi = \psi_\infty \tanh\frac{x}{\sqrt{2}\xi} \text{ where } \psi_\infty^2 = -\frac{\alpha}{\beta} \text{ and } \xi = \frac{\hbar}{\sqrt{2m^*|\alpha|}}$$



Previously, in lecture 2

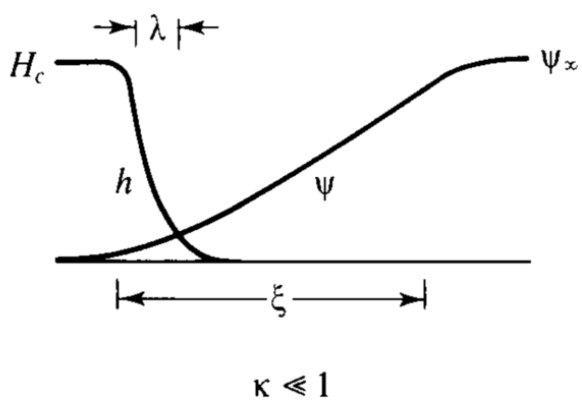
From the Ginzburg-Landau equations

1) Two characteristic lengths in a superconductor

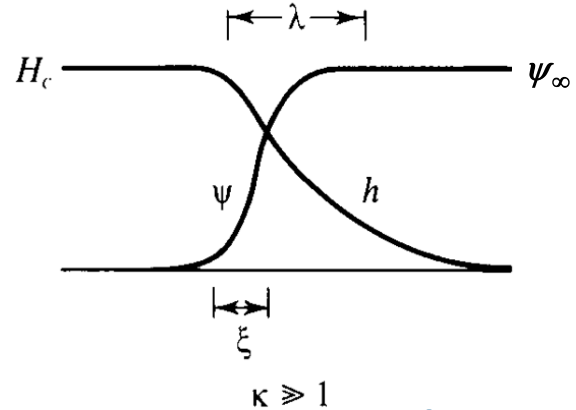
$$\xi = \frac{\hbar}{\sqrt{2m^*|\alpha|}} \quad \text{and} \quad \lambda = \sqrt{\frac{m^* c^2}{4\pi|\psi|^2 e^{*2}}}$$

coherence length — penetration depth

2) Two types of superconductors, depending on $\kappa = \frac{\lambda}{\xi}$



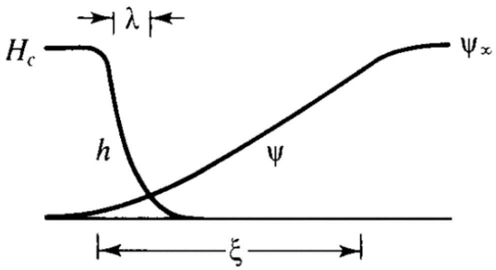
Type-I superconductor



Type-II superconductor

Type-I, type-II and domain-wall energy

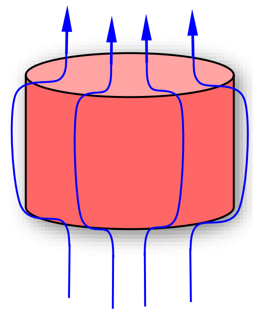
Type-I superconductor



$\kappa \ll 1$

$$\Delta E = A\frac{H_c^2}{8\pi}(\xi - \lambda) > 0$$

$H < H_c$



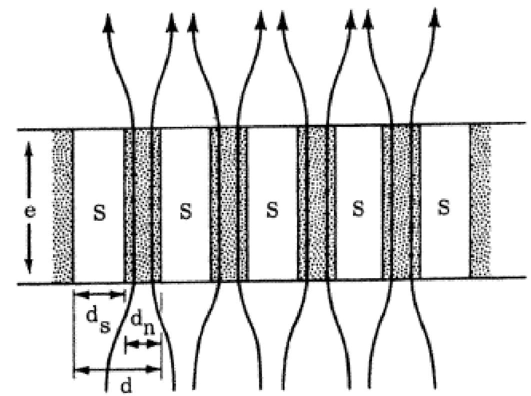
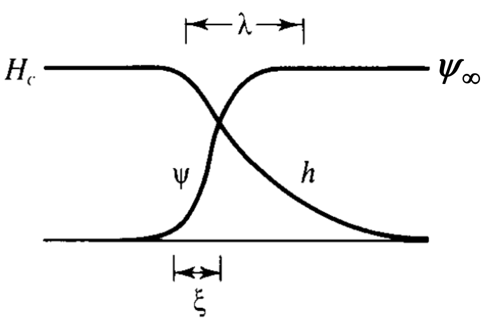


Figure 2-8

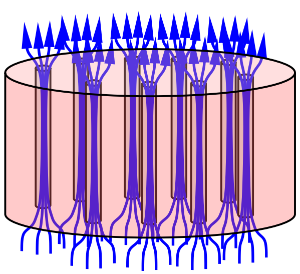
Type-II superconductor

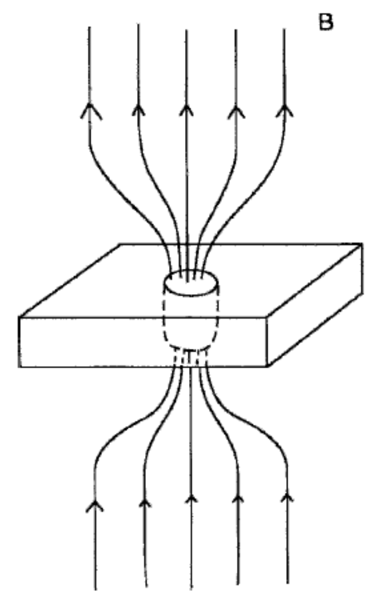


$\kappa \gg 1$

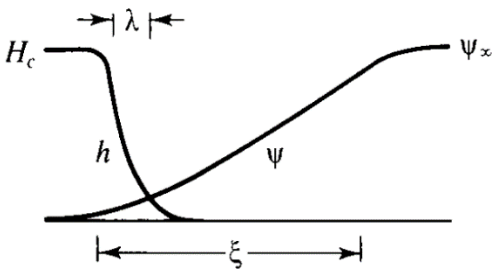
$$\Delta E = A\frac{H_c^2}{8\pi}(\xi - \lambda) < 0$$

$H_{c1} < H < H_{c2}$





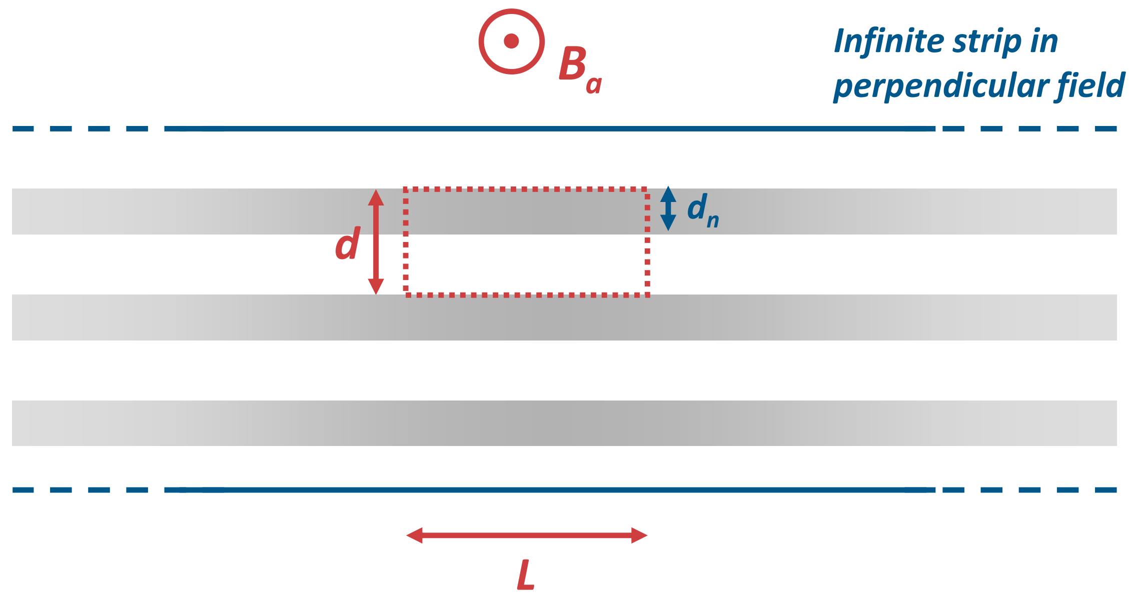
Type-I superconductors
Intermediate state: laminar superconductivity



$\kappa \ll 1$

$$\kappa < \frac{1}{\sqrt{2}} \Rightarrow \Delta E > 0$$

wall energy



Locally the field raises to $B_c \Rightarrow$ superconductivity is suppressed

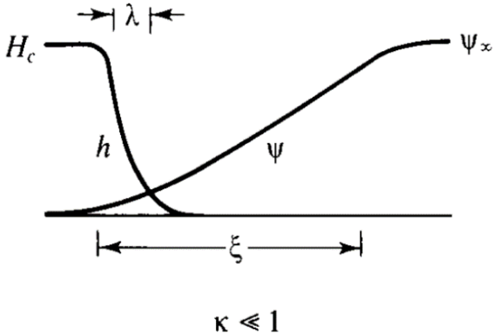
$$\Phi_a = B_a L d = 0 L d_s + B_c L d_n = B_c L d_n$$

$$\boxed{d_n = \frac{B_a}{B_c} d}$$

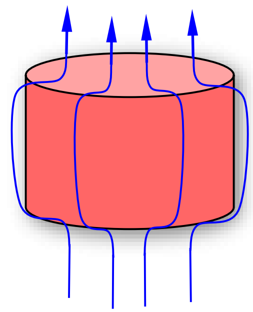
The thinner the film, the smaller the domain repetition distance d (obtained by minimizing the total free energy)

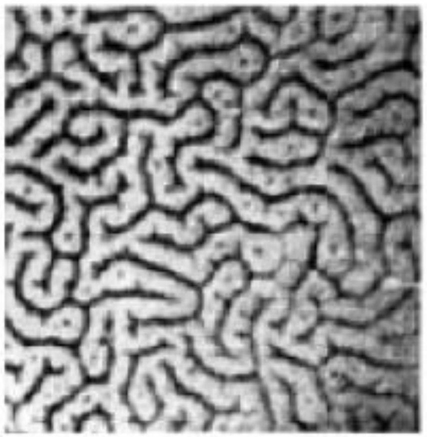
Type-I, type-II and domain-wall energy

Type-I superconductor



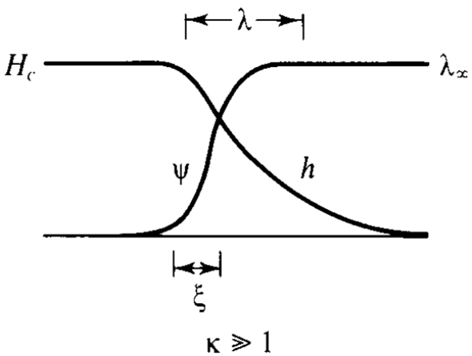
$H < H_c$



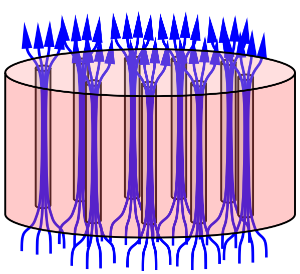


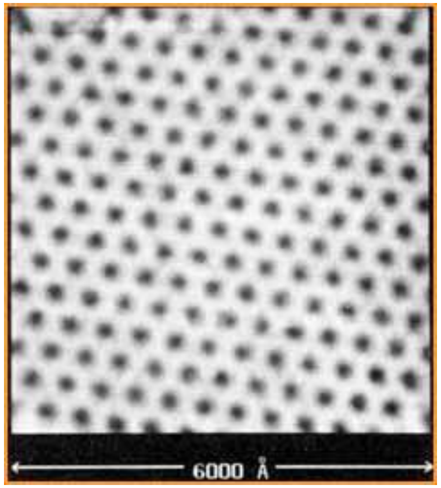
Laminar intermediate state

Type-II superconductor



$H_{c1} < H < H_{c2}$



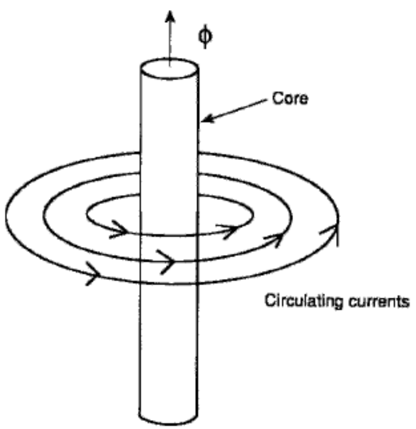


Mixed state

The structure of an isolated vortex

How to modify the 1st London equation in the presence of a vortex

$$\frac{4\pi\lambda^2}{c}\vec{\nabla}\times\vec{j}_s+\vec{h}=\Phi_0\delta(\vec{r})\hat{z}$$



Is it a good choice?

$$\frac{4\pi\lambda^2}{c}\int_S\vec{\nabla}\times\vec{j}_s d\vec{S}+\int_S\vec{h}d\vec{S}=\Phi_0$$

$$\frac{4\pi\lambda^2}{c}\oint_{r>>\lambda}\vec{j}_s d\vec{l}+\int_S\vec{h}d\vec{S}=\Phi_0$$

$$\int_S\vec{h}d\vec{S}=\Phi_0$$ **OK!!**

The structure of an isolated vortex

$$\frac{4\pi\lambda^2}{c}\vec{\nabla}\times\vec{j}_s+\vec{h}=\Phi_0\delta(\vec{r})\hat{z}$$

$$\vec{\nabla}\times\vec{h}=\frac{4\pi}{c}\vec{j}\quad\Rightarrow\quad\lambda^2\vec{\nabla}\times\vec{\nabla}\times\vec{h}+\vec{h}=\Phi_0\delta(\vec{r})\hat{z}$$

Analytical solution

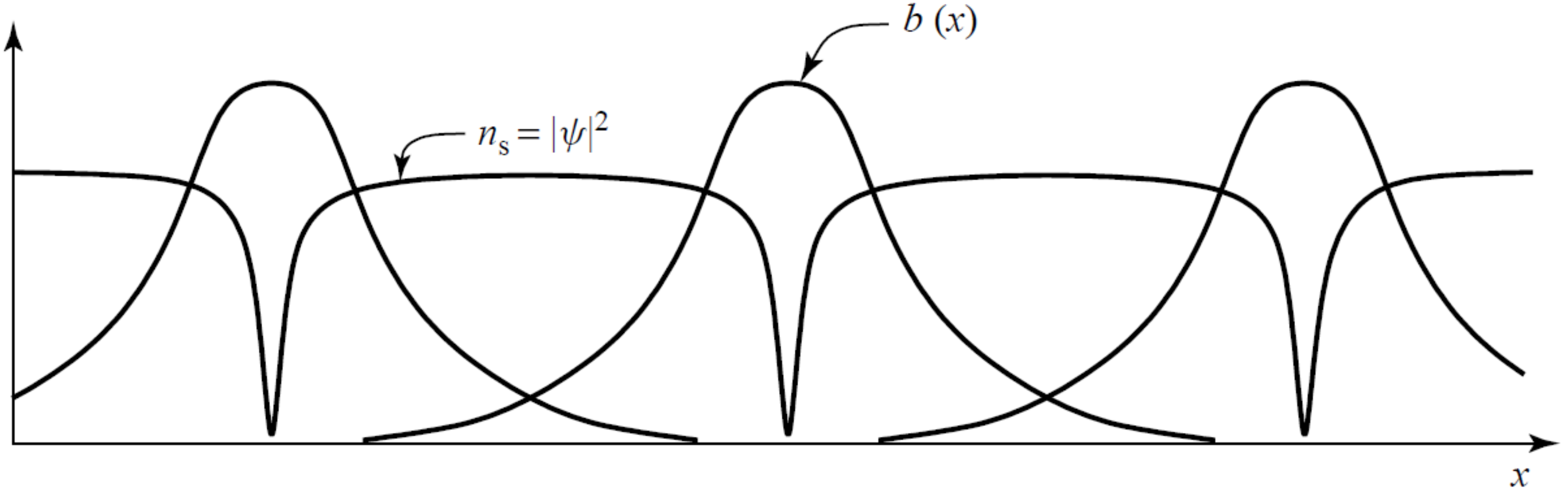
$$h(r)=\frac{\Phi_0}{2\pi\lambda^2}K_0\left(\frac{r}{\lambda}\right)$$

Two limits

$r\to\infty$ $$h(r)\to\frac{\Phi_0}{2\pi\lambda^2}\left(\frac{\pi}{2}\frac{\lambda}{r}\right)^{\frac{1}{2}}e^{-r/\lambda}$$

$\xi<r<\lambda$ $$h(r)\approx\frac{\Phi_0}{2\pi\lambda^2}\left[ln\frac{\lambda}{r}+0.12\right]$$

The structure of an isolated vortex



$$|\psi(r)| = \psi_\infty \tanh \frac{r}{\sqrt{2}\xi}$$

$$h(r) \approx \frac{\Phi_0}{2\pi\lambda^2}\left[\ln\frac{\lambda}{r} + 0.12 \right]$$

Coherence length ξ and Upper critical field H_{c2}

The 1st G-L equation $\alpha\,\psi+\beta\,|\psi|^2\psi+\frac{1}{2m^*}\left(-i\hbar\vec{\nabla}-\frac{e^*}{c}\vec{a}\right)^2\psi=0$

Close to H_{c2} $|\psi|^2$ is small

$$\frac{1}{2m^*}\left(-i\hbar\vec{\nabla}-\frac{e^*}{c}\vec{a}\right)^2\psi=-\alpha\,\psi$$

If H // z, we choose $\vec{a}=(0,Hx,0)$

$$\xi=\frac{\hbar}{\sqrt{2m^*|\alpha|}}$$

$$\left[-\nabla^2+\frac{4\pi i}{\Phi_0}Hx\frac{\partial}{\partial y}+\left(\frac{2\pi H}{\Phi_0}\right)^2x^2\right]\psi=\frac{1}{\xi^2}\psi$$

$$\Phi_0=\frac{hc}{e^*}$$

Other possible choices: $\vec{a}=(-Hy,0,0)$ *and* $\vec{a}=\left(-\frac{H}{2}y,\frac{H}{2}x,0\right)$

Coherence length ξ and Upper critical field H_{c2}

$$\left[-\nabla^2 + \frac{4\pi i}{\Phi_0} H x \frac{\partial}{\partial y} + \left(\frac{2\pi H}{\Phi_0}\right)^2 x^2\right]\psi = \frac{1}{\xi^2}\psi$$

We look for a solution in the form

$$\psi = e^{ik_y y} e^{ik_z z} f(x)$$

$$-f''(x) + \left(\frac{2\pi H}{\Phi_0}\right)^2 (x - x_0)^2 f = \left(\frac{1}{\xi^2} - k_z^2\right) f$$ ***where*** $$x_0 = \frac{k_y \Phi_0}{2\pi H}$$

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + \frac{1}{2} m\omega^2 x^2 \Psi(x) = E\Psi(x)$$

Quantum Harmonic Oscillator: Schrodinger Equation

Coherence length ξ and Upper critical field H_{c2}

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2}+\frac{1}{2}m\omega^2x^2\Psi(x)=E\Psi(x)$$

$$-f''(x)+\left(\frac{2\pi H}{\Phi_0}\right)^2(x-x_0)^2f=\left(\frac{1}{\xi^2}-k_z^2\right)f$$

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega=\left(n+\frac{1}{2}\right)\hbar\left(\frac{e^*H}{m^*c}\right)$$

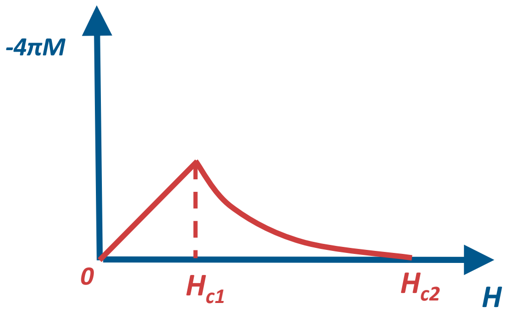
$$\frac{\hbar^2}{2m^*}\left(\frac{1}{\xi^2}-k_z^2\right)=\left(n+\frac{1}{2}\right)\hbar\left(\frac{e^*H}{m^*c}\right)$$

$$H=\frac{\Phi_0}{2\pi(2n+1)}\left(\frac{1}{\xi^2}-k_z^2\right)\Rightarrow\quad H_{c2}=\frac{\Phi_0}{2\pi\xi^2}$$

$$H_{c2}=\sqrt{2}\kappa\,H_c$$

$$\kappa=\frac{\lambda}{\xi}\qquad \xi^2=\frac{\hbar^2}{2m^*|\alpha|}\qquad \lambda^2=\frac{m^*c^2}{4\pi|\psi|^2e^{*2}}\qquad |\psi|^2=\frac{|\alpha|}{\beta}\qquad \frac{\alpha^2}{2\beta}=\frac{H_c^2}{8\pi}\qquad \Phi_0=\frac{hc}{e^*}$$

Lower critical field H_{c1} for $\kappa >> 1$



$$G_S\Big|_{no\ flux} = G_S\Big|_{1st\ vortex}$$

$$G = F - \frac{H\int h dr}{4\pi}$$

$$F_S = F_S + \varepsilon_1 L - \frac{H_{c1}\int h dr}{4\pi} = F_S + \varepsilon_1 L - \frac{H_{c1}\Phi_0 L}{4\pi}$$

Free energy of a vortex line

The lower critical field is $H_{c1} = \dfrac{4\pi\varepsilon_1}{\Phi_0}$

$$H_{c1} = \frac{4\pi\varepsilon_1}{\Phi_0}$$

Lower critical field H_{c1} for $\kappa >> 1$

Free energy of a vortex line

$$\varepsilon_1 = F_m + F_{kin} = \int \frac{h^2}{8\pi} dS + \int \frac{1}{2} m^* v_s^2 n_s dS$$

$$= \frac{1}{8\pi} \int \left[h^2 + \lambda^2 \left| \vec{\nabla} \times \vec{h} \right|^2 \right] dS$$

$$\vec{j}_s = -n_s e^* \vec{v}_s$$

$$\vec{\nabla} \times \vec{h} = \frac{4\pi}{c} \vec{j}$$

$$\lambda^2 = \frac{m^* c^2}{4\pi\, n_s^2 e^{*2}}$$

Integration over $(S - \pi\xi^2)$

Vector identity $\left| \vec{\nabla} \times \vec{h} \right|^2 = \vec{h} \cdot \left[\vec{\nabla} \times \left(\vec{\nabla} \times \vec{h} \right) \right] + \vec{\nabla} \cdot \left[\vec{h} \times \left(\vec{\nabla} \times \vec{h} \right) \right]$

$$\epsilon_1 = \frac{1}{8\pi} \int (\mathbf{h} + \lambda^2 \operatorname{curl} \operatorname{curl} \mathbf{h}) \cdot \mathbf{h}\, dS + \frac{\lambda^2}{8\pi} \oint (\mathbf{h} \times \operatorname{curl} \mathbf{h}) \cdot d\mathbf{s}$$

Lower critical field H_{c1} for $\kappa >> 1$

$$\epsilon_1 = \frac{1}{8\pi}\int(\mathbf{h} + \lambda^2 \,\mathrm{curl}\,\mathrm{curl}\,\mathbf{h}) \cdot \mathbf{h}\, dS + \frac{\lambda^2}{8\pi}\oint(\mathbf{h} \times \mathrm{curl}\,\mathbf{h}) \cdot d\mathbf{s}$$

$$= \frac{1}{8\pi}\int |\mathbf{h}|\Phi_0\delta_2(\mathbf{r})\, dS + \frac{\lambda^2}{8\pi}\oint(\mathbf{h} \times \mathrm{curl}\,\mathbf{h}) \cdot d\mathbf{s}$$

Integration over $(S-\pi\xi^2)$

It follows $\varepsilon_1 = \frac{\lambda^2}{8\pi}\left[h\frac{dh}{dr}2\pi r\right]_\xi$ where $h(r) \approx \frac{\Phi_0}{2\pi\lambda^2} \ln\frac{\lambda}{r}$

$$\varepsilon_1 \approx \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \ln\kappa$$

Lower critical field H_{c1} for $\kappa >> 1$

$$\varepsilon_1 \approx \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \ln\kappa$$

$$H_{c1} = \frac{4\pi\varepsilon_1}{\Phi_0} \approx \frac{\Phi_0}{4\pi\lambda^2}\ln\kappa \quad = \frac{H_c}{\sqrt{2}\kappa}\ln\kappa$$

$$\Phi_0 = \frac{hc}{e^*}$$

$$\frac{\alpha^2}{2\beta} = \frac{H_c^2}{8\pi}$$

$$|\psi|^2 = \frac{|\alpha|}{\beta}$$

$$\xi^2 = \frac{\hbar^2}{2m^*|\alpha|}$$

H_c , H_{c1} and H_{c2}

$$H_{c1} = \frac{\Phi_0}{4\pi\lambda^2} ln\kappa \qquad H_{c2} = \frac{\Phi_0}{2\pi\xi^2}$$

$$H_{c1} = \frac{H_c}{\sqrt{2}\kappa} ln\kappa \qquad H_{c2} = \sqrt{2}\kappa \, H_c$$

1957: BCS - Microscopic theory of superconductivity

1972

In 1957, Bardeen, Cooper, and Schrieffer propose their microscopic theory of superconductivity, the BCS theory.

Bardeen, Cooper and Schrieffer

- *Electron-phonon coupling as origin of attractive interaction between electrons*
- *Cooper Pairs: paired electrons with opposite momentum and spin (bosons with zero spin) forming a macroscopic condensed state*
- *The binding energy introduces an energy gap Δ between paired and unpaired state*

In 1959, Gorkov showed that the macroscopic GL theory can be derived from microscopic BCS theory at temperatures close to the critical one

$$e^* = 2e \qquad m^* = 2m_e \qquad |\psi| = \Delta$$

The Abrikosov vortex lattice

As shown for the determination of H_{c2}, close to H_{c2} and for H // z, the 1st G-L equation

$$\alpha\,\psi + \beta\,|\psi|^2\psi + \frac{1}{2m^*}\left(-i\hbar\vec{\nabla} - \frac{e^*}{c}\vec{a}\right)^2\psi = 0$$

can be rewritten as

$$\left[-\nabla^2 + \frac{4\pi i}{\Phi_0}Hx\frac{\partial}{\partial y} + \left(\frac{2\pi H}{\Phi_0}\right)^2 x^2\right]\psi = \frac{1}{\xi^2}\psi$$

Solutions can have the form

$$\psi = e^{ik_y y}e^{ik_z z}f(x)$$

$$-f''(x) + \left(\frac{2\pi H}{\Phi_0}\right)^2(x-x_0)^2 f = \left(\frac{1}{\xi^2} - k_z^2\right)f \qquad \textit{where} \quad x_0 = \frac{k_y\Phi_0}{2\pi H}$$

$$\psi = e^{ik_y y} e^{ik_z z} f(x)$$

The Abrikosov vortex lattice

$$\left[-\nabla^2 + \frac{4\pi i}{\Phi_0} H x \frac{\partial}{\partial y} + \left(\frac{2\pi H}{\Phi_0} \right)^2 x^2 \right] \psi = \frac{1}{\xi^2} \psi$$

becomes

$$-f''(x) + \left(\frac{2\pi H}{\Phi_0} \right)^2 (x - x_k)^2 f = \left(\frac{1}{\xi^2} - k_z^2 \right) f \quad \text{where} \quad x_k = \frac{k_y \Phi_0}{2\pi H}$$

From slide 15, at $H = H_{c2}$ it is $k_z = 0$

It follows

$$\psi_k = e^{iky} f(x) = \exp(iky) \exp\left[-\frac{(x - x_k)^2}{2\xi^2} \right] \quad \text{where } k = k_y$$

$$\psi_L = \int dk \, g(k) \psi_k$$

The Abrikosov vortex lattice

We expect a crystalline array of vortices to have lower energy than a random one

Therefore we restrict the values of k to

$$k_n = nq \qquad \psi_L = \sum_n c_n \psi_n$$

This choice determines a periodicity in y and x

$$\Delta y = \frac{2\pi}{q}$$

$$\psi_L(x, y+\Delta y) = \psi_L(x, y)$$

$$\psi_L(x+\Delta x, y) = \psi_L(x, y)$$

$$x_n = \frac{nq\Phi_0}{2\pi H} \Rightarrow \Delta x = \frac{q\Phi_0}{2\pi H} = \frac{\Phi_0}{H\Delta y}$$

And it follows

$$H\Delta x\Delta y = \Phi_0$$

A.A. Abrikosov, Sov. Phys. JETP <u>5</u> (1957) 1174

The Abrikosov vortex lattice

The solution

$$\psi_k = e^{iky} f(x) = \exp(iky)\exp\left[-\frac{(x-x_k)^2}{2\xi^2}\right]$$

can be generalized as the overlap of periodic functions

$$\psi_L = \sum_n C_n \psi_n = \sum_n C_n \exp(inqy)\exp\left[-\frac{(x-x_n)^2}{2\xi^2}\right]$$

For the periodicity in x, we impose conditions on C_n

$$C_n = C_{n+\nu}$$

Square lattice $\nu = 1$

Triangular lattice $\nu = 2$ and $C_1 = iC_0$

The Abrikosov vortex lattice

To determine the shape of the vortex lattice for $H < H_{c2}$ the non-linear term in the 1st G-L equation cannot be neglected

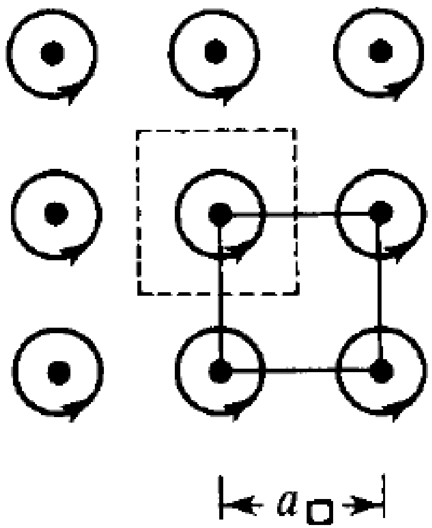
Abrikosov has shown that the solution depends on

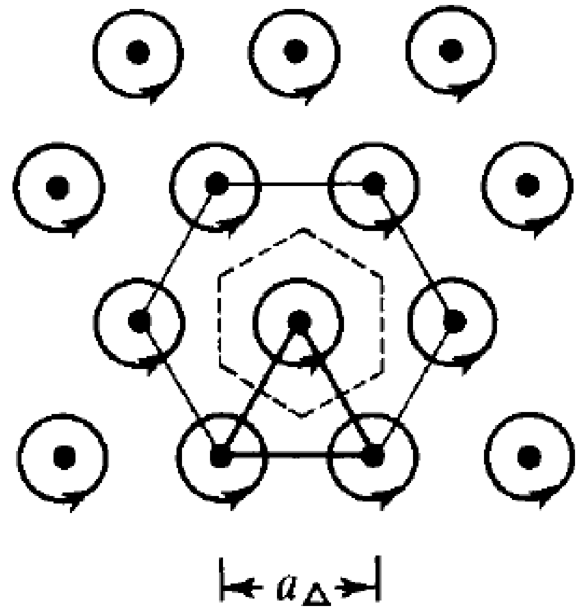
$$\beta_A \equiv \frac{\left\langle \psi_L^4 \right\rangle}{\left\langle \psi_L^2 \right\rangle^2}$$

For the square lattice $\beta_A = 1.18$ and the lattice spacing is $a_{\square} = \left(\frac{\Phi_0}{B} \right)^{1/2}$

For the triangular lattice $\beta_A = 1.16$ and the lattice spacing is $a_{\Delta} = \left(\frac{4}{3} \right)^{1/4} \left(\frac{\Phi_0}{B} \right)^{1/2}$

The Abrikosov vortex lattice





$$a_{\square} < a_{\Delta}$$

$$a_{\square} = \left(\frac{\Phi_0}{B}\right)^{1/2}$$

$$a_{\Delta} = \left(\frac{4}{3}\right)^{1/4}\left(\frac{\Phi_0}{B}\right)^{1/2}$$

Interaction between vortices

From slide 18, the free energy of a single vortex is

$$\varepsilon_{1-vortex} \approx \frac{\Phi_0}{8\pi} h(0)$$

In the case of 2 vortices

$$\varepsilon_{2-vortices} = \frac{\Phi_0}{8\pi}\left[h_1(r_1) + h_1(r_2) + h_2(r_1) + h_2(r_2)\right]$$

$$= 2\left[\frac{\Phi_0}{8\pi} h_1(r_1)\right] + \frac{\Phi_0}{4\pi} h_1(r_2)$$

$$= 2\varepsilon_{1-vortex} + \varepsilon_{int\,eraction}$$

Interaction between vortices

$$\varepsilon_{interaction} = \frac{\Phi_0}{4\pi} h_1(r_2) \quad where \quad h(r) = \frac{\Phi_0}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right)$$

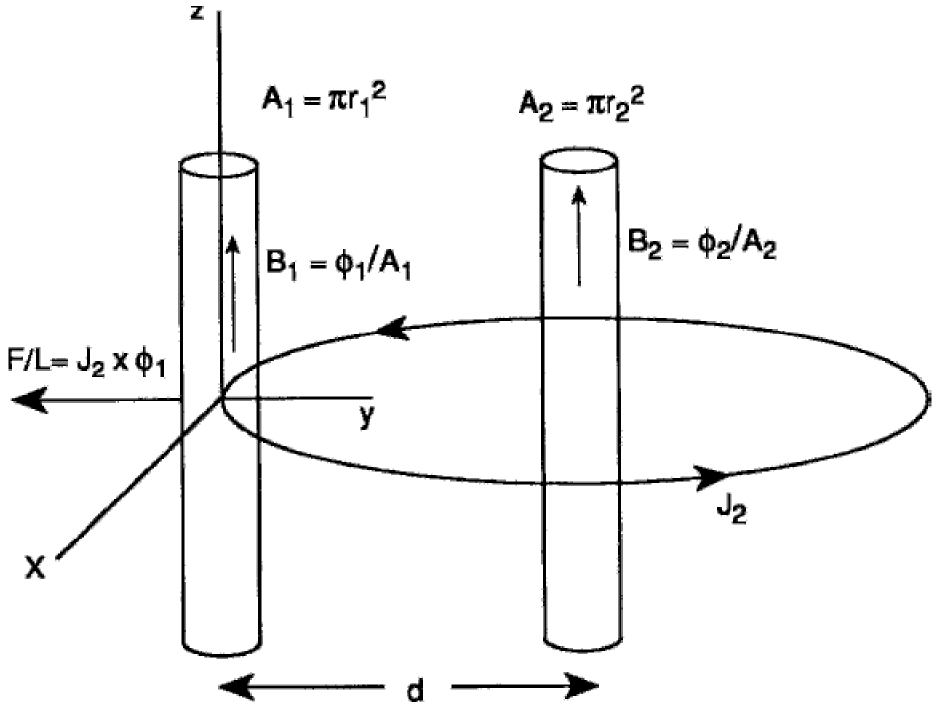
$$= \frac{\Phi_0^2}{8\pi^2\lambda^2} K_0\left(\frac{r_{1-2}}{\lambda}\right)$$

And the force is

$$f_{2x} = -\frac{\partial}{\partial x_2}\varepsilon_{interaction}$$

$$= -\frac{\Phi_0}{4\pi}\frac{\partial h_1(r_2)}{\partial x_2}$$

$$= \frac{\Phi_0}{c} J_{1y}(r_2)$$



Interaction between vortices

The force of vortex 1 on vortex 2 is

$$f_2 = J_1(r_2) \times \frac{\Phi_0}{c}$$

The obvious generalization to an arbitrary array is

$$f = J_s \times \frac{\Phi_0}{c}$$

J_s is the total supercurrent due to all other vortices J_{array} + any transport current J_{ext} at the vortex core position.

Obviously, at equilibrium

$$J_{array} \times \frac{\Phi_0}{c} = 0$$

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